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Modeling Decision Making

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1 Introduction

Quantum-like modeling of decision making was triggered by three types of psychological experiments that showed the conjunction and disjunction fallacies and the order effect, challenging the application of classical probability theory to cognition [1, 2, 3]. Quantum probability theory was found to be able to account for such phenomena [4, 5, 6]. The essential mathematical aspects of quantum theory that came in handy were the representation of states by vectors in a Hilbert space rather than by points in phase space and the generally non-commuting structure of the operators acting on such states. However, quantum theory was developed to deal with microscopic entities like electrons and atoms that showed coherence effects (such as interference and entanglement) absent in macroscopic objects like chairs and tables. It is far from clear how the human brain, a hot and noisy macroscopic system, can act as if it were a quantum information processor. Quantum information processing is possible only if one uses *qubits* or quantum bits. A qubit is the basic unit of quantum information. Unlike the classical binary bit physically realized with a two-state (on-off) device, a qubit is a two-state quantum mechanical system. It is the simplest quantum system displaying the peculiarity of quantum mechanics, namely the superposition of two orthogonal states. One can also have qutrits and other multiple state systems. These provide an inherent parallel processing capacity to quantum computers which classical computers do not have. This should enable them to solve certain problems much faster than any classical computer using the best currently known algorithms.

Hence, if the information processings carried out by biological, social or financial systems show some essentially quantum-like behaviour, the systems must necessarily be quantum mechanical themselves. But there is no compelling evidence of that as yet. On the contrary, all

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evidence points to the fact that macroscopic systems are classical. Quantum mechanics itself leads to the conclusion that an approximately macroscopic world emerges as a consequence of decoherence [7]. Decoherence is the absence of superposition of macroscopic states, and it is attributed to interactions between quantum systems and the larger macroscopic environment from which they can never be completely isolated. Hence, the assumption that inherently macroscopic entities like biological, economic and financial systems behave in some ways like quantum systems is difficult to justify.

There is another problem too with quantum-like modeling of cognition, namely that it is afflicted with the same controversies over interpretations as quantum mechanics, arising fundamentally from the measurement problem which remains unsolved even after nearly a century of incessant attempts to solve it [8].

Clearly, an alternative theoretical structure that does not suffer from the above difficulties would be preferable. The main purpose of this paper is to suggest that such an alternative indeed exists in classical optical theory.

The two main features of quantum mechanics that make it attractive to model non-classical cognitive behaviour like the conjunction and disjunction fallacies and the order effect are (a) coherence (exemplified by interference and entanglement phenomena) and (b) the occurrence of non-commuting observables. However, coherence in classical optics has been known since the days of Thomas Young and Augustin-Jean Fresnel in early nineteenth century, and consequently, like quantum mechanical states, classical optical states can also be described by vectors in Hilbert spaces. In the last couple of decades even entanglement and Bell violations have been observed with classical optical beams [9, 10]. Hence, there is a close similarity between the mathematical structures of classical optics and quantum mechanics [11]. Furthermore, the famous Poincaré sphere representation of polarization states in optics is spherically symmetric, making it invariant under the orthogonal transformation group $O(3)$ which in non-Abelian. Representing cognitive states by the states on a Poincaré-like sphere therefore enables the order effect also to be accounted for without necessarily invoking quantum mechanical non-commuting operators.

Finally, it was Niels Bohr who first recognized that contextuality and complementarity which play important roles in quantum mechanics are also inherent in the wider field of psychology and human knowledge [12]. This aspect of human knowledge is at odds with any two-valued logic such as Aristotelian logic that underpins classical probability theory. Interestingly, many-valued logic systems existed in ancient India. Particular mention must be made of the *chatuṣkoṭi* system of Buddhist logic and the *syādvāda* system of Jaina logic which can be formalized as paraconsistent systems [13]. In the *syādvāda* system, apart from true and false propositions, there is a third kind of valid proposition called *avaktavyam*, the *unspeakable*, and all of them as well as their four combinations are *contextual*. In other words, all valid propositions are contextual and there can be only seven of them. That is why it is known as *saptabhanginaya*, seven-fold predication. Hence, many-valued logics such as *syādvāda* provide an appropriate philosophical foundation for mathematically modelling cognitive phenomena which violate classical probability theory based on two-valued logic.

The essence of this logic system is captured by the parable of a group of blind men and an elephant. Seven blind men who have never come across an elephant before learn and imagine what it is like by touching it. Each blind man touches a different part of the elephant's body such as the side, ear or the tusk. They then describe the elephant to each other based on their limited experience of it, and ssuspecting the others to be dishonest, come to blows. The moral is that humans have a tendency to claim absolute truth based on their limited experience, ignoring other people's experiences which may also be equally valid. In cognitive modeling lived experience, which is necessarily contextual and limited, must play a central role.

2 Contextuality, Logic and Probability

2.1 Contextuality

Contextuality is inherent in cognition and can be of many kinds. Five types of contextuality have been considered so far in relation to cognitive modelling: the contextualities (i) of truth, (ii) of being, (iii) of meaning, (iv) in physics and (v) contextuality by default [14]. That the truth of a sentence and the meaning of a word depend on the context in which they occur is quite obvious. Contextuality of being arose out of Heidegger's concept of *Dasein* which is of a being in practical engagement with an environment. It is related to contemporary issues of AI [15].

Contextuality by Default (CbD) is a new theory of contextuality which rejects the general view that 'everything depends on everything else' by a specific tenet, namely that the identity of a 'random variable' representing a decision maker's response is determined not only by its *content* but also by the *context*, the systematically recorded conditions under which the variable is observed [16, 17, 18].

Contextuality in physics has been defined in two essentially different ways, one within quantum mechanics itself and the other within hidden variable interpretations of quantum mechanics. In the first type defined by Bohr, measurement results depend on experimental set ups, mutually exclusive experimental set ups giving rise to mutually exclusive phenomena which are reconciled within the over arching Principle of Complementarity. However, following the Kochen-Specker theorem concerning hidden variables [19], quantum contextuality can also be defined as dependence of a measurement result of a quantum observable on which other commuting observables are within the same measurement set.

There are two other types of contextuality relevant for cognition, namely contextuality of *saṃskāra* and contextuality of scale. That which determines the disposition of a person is called *saṃskāra* in Sanskrit, and this disposition changes every time a new act of cognition occurs, leaving behind an imprint and changing the *saṃskāra*. Every person has her own characteristic *saṃskāra*, and no two persons have exactly the same *saṃskāra*. This results in different dispositions leading to different decisions even in almost identical circumstances. The contextuality of scale can be understood with the example of a rose petal. It is a symbol of beauty and love. Imagine looking at it through a series of ever more powerful microscopes. The petal first dissolves into its internal structure of cells, then the atoms and molecules which make up the cells, and finally the elementary particles which make up the atoms and molecules. At the opposite end, think of moving away from the petal until it disappears from our view. The rose petal is a rose petal only when viewed from the human scale, not at every scale.

2.2 Syādvāda Logic

This system of logic has three basic values: 'true', 'false' and '*avaktavyam*'. The Sanskrit word *avaktavyam* means 'not expressible in words or language', i.e. *unspeakable*. This leads to a seven-fold predication known as *saptabhanginaya*. The system is intrinsically *contextual*. The three basic truth values are written as (i) *syād asti* (*asti* = true), (ii) *syād nāsti* (*nāsti* = *na* + *asti* = false) and (iii) *syād avaktavyam*. The Sanskrit word *syād* has been variously translated as 'perhaps', 'may be', 'in some way', 'conditionally', 'from a certain perspective' and so on. The most appropriate translation for our purpose would be 'under certain conditions' or 'in a certain context'. This is supported by the typical example given, namely if not baked, a clay pot is black; if baked, it is red; and during the baking process its colour is *unspeakable* (*avaktavyam*).

The other four compounds are (iv) *syād asti cha nāsti cha* (*syād* true and *syād* false), (v) *syād asti cha avaktavyam cha* (*syād* true and *syād* unspeakable), (vi) *syād nāsti cha avaktavyam cha* (*syād* false and *syād* unspeakable), and (vii) *syād asti cha nāsti cha avaktavyam cha* (*syād* true and *syād* false and *syād* unspeakable). The Sanskrit word *cha* means 'and'.

Using the quantifier $\forall$ the first three can be written as (i) $\forall x\,[\phi(x) \to p(x)]$; (ii) $\forall x\,[\phi(x) \to \neg p(x)]$; (iii) $\forall x\,[\phi(x) \to q(x)]$, x standing for a variable (a placeholder) which ranges over the domain of pots, ϕ for a well formed formula that specifies some condition (like for example 'baked'), p for some predicate (like say 'red') and q for the predicate *avaktavyam*. An 'example' of the first of these three in plain English would be: for all x (say clay pots) the condition $\phi(x)$ (say 'baked') implies that the pot is red.

The other four compounds may be written as

(iv) $\forall x\,[\phi(x) \to p(x) \wedge \phi'(x) \to \neg p(x)] \wedge \neg[\phi(x) \leftrightarrow \phi'(x)]$,

(v) $\forall x\,[\phi(x) \to p(x) \wedge \phi'(x) \to q(x)] \wedge \neg[\phi(x) \leftrightarrow \phi'(x)]$,

(vi) $\forall x\,[\phi(x) \to \neg p(x) \wedge \phi'(x) \to q(x)] \wedge \neg[\phi(x) \leftrightarrow \phi'(x)]$,

(vii) $\forall x\,[\phi(x) \to p(x) \wedge \phi'(x) \to \neg p(x) \wedge \phi''(x) \to q(x)] \wedge \neg[\phi(x) \leftrightarrow \phi'(x)] \wedge \neg[\phi'(x) \leftrightarrow \phi''(x)] \wedge \neg[\phi(x) \leftrightarrow \phi''(x)]$.

Written in this formal way, the seven predications are self-consistent as they hold under *mutually exclusive* conditions.

2.3 Probability

Logic and probability theory are related. P. C. Mahalanobis gives the example of the third truth value *avaktavyam* or *unspeakable* in *syādvāda* as the conceptual origin of probability, the uspeakable resolving into the speakable p and $\neg p$ with certain probabilities [20]. Without the third value the origin of probabilities remains obscure.

There are two types of probability that are predominantly used, Bayesian and frequentist. Bayesian probability is probability as a 'degree of belief'. It was introduced by Thomas Bayes and subsequently promoted by Pierre-Simon Laplace and others. It was the mainstay for more than a century before mathematicians introduced the frequentist idea based on counting the occurrences of events to inject more objectivity into the notion of probability than the subjective notion of a 'degree of belief' allowed. However, the pendulum started to swing back to the older Bayesian idea by the middle of the 20th century, mainly because of the large data bases created and the need for statistical analysis. Bayesian probability is again of two types, subjective and objective. Bayes himself as well as Laplace were 'objectivists'. The person who introduced a strictly subjective interpretation of Bayesian probability theory was De Finetti [21, 22]. He viewed science as a human activity, a product of thought, having probability as its main tool. To quote De Finnetti,

'...no science will permit us say: this fact will come about, it will be thus and so because it follows from a certain law, and that law is an absolute truth. Still less will it lead us to conclude skeptically: the absolute truth does not exist, and so this fact might or might not come about, it may go like this or in a totally different way, I know nothing about it. What we can say is this: I foresee that such a fact will come about, and that it will happen in such and such a way, *because past experience and its scientific elaboration by human thought make this forecast seem reasonable to me.* (italics added) (as quoted in [22]).

With this in mind, Bayes' definition of probability goes like this–an agent's assignment of probability p for the occurrence of an event E means that the agent is willing to bet any amount up to p dollars for a coupon worth one rupee if E happens. Conversely, the agent is willing to sell the coupon for any amount from p dollars and up. The probability turns out to lie between 0 and 1, as in the frequentist case. But the difference is profound–decisions for future actions are all based on 'degrees of belief', like the decision to take an umbrella along if the forecast for rain is, say 70%.

When the degree of belief changes due to new information, the Bayesian probability changes too. This crucially differentiates Bayesian probability from frequentist probability which is cast in stone. Furthermore, the frequentist interpretation applies only to multiple trials. It is an 'ensemble property' and nothing is said about single cases or individual events. It is in fact impossible to make any valid probabilistic prediction regarding the outcome of a single trial from the frequentist principle. As put by Appleby [23],

'At the point of empirical application every piece of predictive probabilistic reasoning presents us with a dilemma of the following general form "Given that the probability of X is p are we, or are we not prepared to bet that X will in fact happen, in a single trial?" ...

Making the best decision in the face of uncertainty–calculating the best bet– is what probability is for. However distasteful it may be to objectivist-minded philosophers, gambling is in fact the point. Remove the gambling element–remove the concept of a single-case probability–and you remove with it all the empirical applications. What remains is not really probability at all, but abstract measure theory.'

We suggest that subjective Bayesian probability is more appropriate for decision modelling than frequentist probability.

3 Classical Optical Modeling

It so happens that the methods of classical optics and quantum mechanics are amazingly similar. It is on account of the fact that both use Hilbert spaces to represent states. The only difference is in the use of non-commuting operators to represent observables in quantum mechanics. There are no such operators in classical optics. Nevertheless, it is remarkable that quitessentially quantum predictions like the Casimir Effect, anti-coincidence on a beam-splitter as well as entanglement and Bell violations can be reproduced in classical optics by introducing only one additional assumption, namely that there is a classical zero-point field with an energy density per unit frequency interval of $\rho = \hbar\omega^3/(2\pi^2c^3)$ whose scale is set by $\hbar$, the reduced Planck constant. This is known as *stochastic electrodynamics*. It is a relativistic theory of point charges and electromagnetic fields based on three fundamental ideas:

1) The electromagnetic fields satisfy Maxwell's equations with sources given by point charged particles.

2) The particles experience forces due to electromagnetic fields and move according to Newton's second law.

3) The solution of the source-free Maxwell equations is given by a classical electromagnetic zero-point radiation with a Lorentz-invariant spectrum of random classical radiation with an energy per normal mode $E(\omega) = \hbar\omega/2$. A further assumption that is made is that photon detectors have an intensity threshold just above the level of this noise, thus detecting only signals.

A survey of stochastic electrodynamics will be found in Boyer [24] and a local realistic analysis of optical tests of Bell inequalities in Marshall and Santos [25].

Interference

In classical optics a state is represented by a vector in a complex linear vector space. This has the advantage that it is possible to add two vectors to get a third vector. This addition of vectors in a complex linear vector space is precisely the superposition principle we are looking for to account for interference effects.

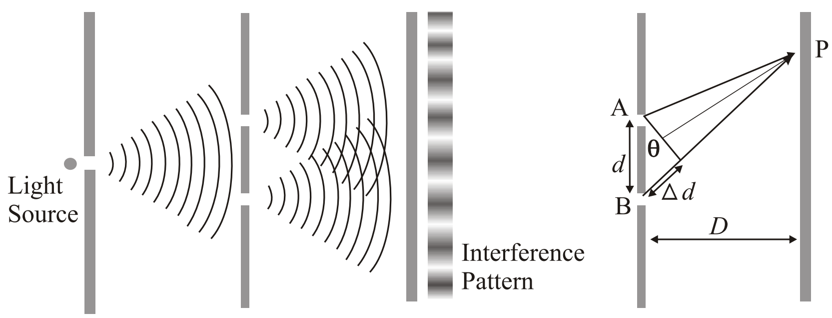

Figure 1: Young's double-slit experiment.

Let us now consider Young's famous double-slit experiment with classical light. Let there be two narrow slits A and B, separated by a distance small compared with the wavelength of a coherent monochromatic light wave which passes through them. After passing through the slits the waves diffract and spread out on the other side (as shown on the left hand side in Fig. 1), overlap and finally arrive at a screen at a distance D. Let us denote the states of the waves emanating from the slits and arriving at a point P on the screen by $|\psi_A\rangle$ and $|\psi_B\rangle$. Then the resultant vector at P is

$$|\Psi\rangle_P = |\psi_A\rangle + e^{i\theta}|\psi_B\rangle \tag{1}$$

where θ is the phase difference between the two waves which travel different optical distances from the slits to arrive at P. Let us now define a complex classical wave function or wave amplitude $\psi(x)$ (x standing for a vector $\vec{x}$) by taking the projection of the vector $|\psi\rangle$ on a 3-dimensional coordinate space: $\psi(x) = \langle x|\psi\rangle$. Then, the wave amplitudes are given by

$$\psi(x)_P = \psi(x)_A + e^{i\theta}\psi(x)_B \tag{2}$$

and the intensity of the resultant wave on the screen is given by

$$I = ||\psi(x)_P||^2 = ||\psi(x)_A||^2 + ||\psi(x)_B||^2 + ||\psi(x)_A||||\psi(x)_B|| \left(e^{i\theta} + e^{-i\theta}\right) \tag{3}$$

$$= ||\psi(x)_A||^2 + ||\psi(x)_B||^2 + 2||\psi(x)_A||||\psi(x)_B|| \cos\theta \tag{4}$$

This shows that as the point P varies along the screen D, the intensity of the light varies sinusoidally. The first two terms determine the constant background intensity and the third term describes the interference pattern. Dividing throughout by the total intensity, one obtains

$$1 = p_A + p_B + 2\sqrt{p_A p_B}\cos\theta. \tag{5}$$

Since p_A and p_B are positive fractions, they can be interpreted as probabilities using the Born rule which is also used in quantum mechanics to interpret Schrödinger wave functions as probability amplitudes. The expression (5) is then the *law of total probability*, exactly as in quantum mechanics. Notice that according to classical probability theory, the total probability p should be given by $p = p_A + p_B = 1$. The law of total probability in classical optics, as well as in quantum mechanics, has an additional term, the 'interference term'. This holds in every linear wave theory after application of the Born rule.

Entanglement

One of the interesting properties of light is that it can be *polarized*. The polarization is conventionally defined by the direction of the electric field vector. In the case of polarized light beams, the beam cross-sections have *uniform* polarization, i.e. the same polarization everywhere. However, this is not generally true. Over the last couple of decades states of classical light have been produced with cross-sections that are not uniformly polarized. As is evident from Fig.2, the electric vector (shown by the red arrows) is not uniform over the beam cross-section. These are examples of radially, azimuthally and spirally polarized light. Such light is entangled in the sense that the beam has no definite polarization–the polarization is different at different positions of the beam cross-section, and hence polarization and position cannot be factored, i.e. they are 'nonseparable' or entangled. Non-separability is the most general feature of entanglement in quantum mechanics too. Entangled classical light has been shown to violate Bell-like inequalities [10, 11], just like entangled quantum states. *Entangled classical light is thus mathematically analogous to quantum entangled states.* However, classical light being completely described by Maxwell's theory which forms the basis of Einstein's special theory of relativity, there is no 'nonlocality' in classical optics, though nonlocality is claimed to be the quintessential feature of quantum entanglement.

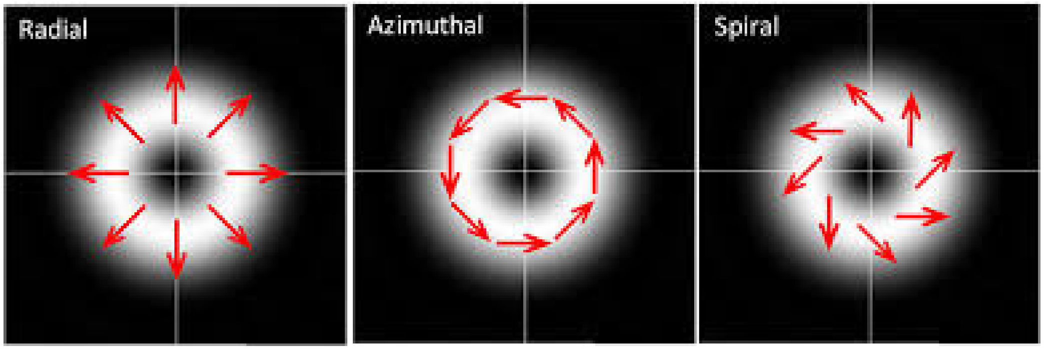

Figure 2: Radially, azimuthally and spirally polarized entangled light.

4 Concluding Remarks

It is thus clear that classical stochastic optics is mathematically completely analogous to quantum mechanics, and hence classical optical modeling (COM) retains all advantages of quantum-like modeling of cognition. It has the added advantage over quantum-like modeling of being free of the interpretational problems of quantum mechanics as well as of the problems of decoherence and nonlocality. It is however different from quantum-like modelling in two ways. First, it takes as comprehensive a view of cognition as possible before attempting to model it mathematically. In this regard use is made of a formalized version of the Jaina many-valued logic system known as *syādvāda* which is intrinsically contextual together with a subjective Bayesian approach to probability to reflect the first-person nature of consciousness and cognition.

There have been many successful applications of quantum-like modelling of cognition to social sciences such as in strategic organizational changes, financial markets which operate under ambiguity and uncertainty, capital formation in economics and public policy analysis [26, 27, 28, 29, 30]. The latest *Human Development Report* (HDR 2022) by UNDP is a significant shift from the standard way of policy thinking. It places central importance on cognitive science to understand decision making in this world of a 'complex of uncertainties'. The standard neoclassical approach is deeply limited in addressing such issues. If a suitable global policy framework is to be developed addressing such uncertainties, new approaches need to be developed. The framework we suggest in this paper is a fairly comprehensive attempt in that direction, emphasizing the role of contextuality in cognition and decision making, implying an underlying non-Boolean many-valued logical structure.

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References

[1] D. Aerts, 'Quantum Structure in Cognition', *J of Math Psych* **53**, 314-348 (2009).

[2] A. Tversky and D. Kahneman, 'Judgements of and by representativeness' (Ch 6) in *Judgement under Uncertainty: Heuristics and Biases*, Cambridge University Press (1982).

[3] D. N. Osherson & E. E. Smith, 'On the adequacy of prototype theory as a theory of concepts', *Cognition* **9**, 35-58 (1981).

[4] D. Aerts, S. Aerts & L. Gabora, 'Experimental Evidence for Quantum Structure in Cognition', *Int Symp on Quantum Interaction QI 2009: Quantum Interaction* pp 59-70 (2009).

[5] E. M. Pothos and J. R. Busemeyer, 'A quantum probability explanation for violations of 'rational' decision theory', *Proc. R Soc B* **276**, 2171-8 (2009).

[6] E. Haven and A. Khrennikov, *Quantum Social Science*, Cambridge University Press (2013).

[7] W. H. Zurek, 'Decoherence and the Transition from Quantum to Classical–Revisited', Los Alamos Science **27**, (2002).

[8] A. Khrennikov,

[9] J. H. Eberly et al, 'Quantum and classical optics–emerging links', *Phys. Scr.* **91**, 063003 (2016) and re

[10] Y. Shen and C. Rosales-Guzmán, 'Nonseparable States of Light: From Quantum to Classical', *Laser Photonics Rev* **16**, 2100533 (2022).

[11] A. Forbes, A. Aiello and B. Ndagano, 'Classically Entangled Light', *Prog. in Optics* **64**, 99-153 (2019).

[12] N. Bohr, *Atomic Physics and Human Knowledge*, Dover Publications Inc.; Illustrated edition (18 November 2010).

[13] G. Priest, 'Coda: Jaina Logic' in *The Fifth Corner of Four: An Essay on Buddhist Metaphysics and the Catuskoti*, Oxford University Press, 2022.

[14] Y. Murayama,' Quantum contextuality and cognitive contextuality: The significance of violations of Bell-type inequalities', *Biosystems* **208**, 104472 (2021)

[15] H. L. Dreyfus, 'Why Heideggerian AI Failed and How Fixing it Would Require Making it More Heideggerian', *Philosophical Psychology* **20** (2), 247-268 (2007).

[16] E. N. Dzhafarov, *Contents, Contexts, and Basics of Contextuality*, arXiv:2103.07954 [math.PR] (2021).

[17] E. N. Dzhafarov, 'Assumption-Free Derivation of the Bell-Type Criteria of Contextuality/Nonlocality', *Entropy* **23** (11), 1543 (2021).

[18] I. Basieva, V. H. Cervantes, E. N. Dzhafarov A. Khrennikov, *True Contextuality Beats Direct Influences in Human Decision Making*, arXiv: 1807.05684 [q-bio. NC] (2019).

[19] S. Kochen and E. Specker, 'The Problem of Hidden Variables in Quantum Mechanics', *Journal of Mathematics and Mechanics* **17**, 59-87 (1967).

[20] P. C. Mahalanobis, 'The foundations of statistics', *Dialectica* **8**(2), 95-111 (1954), reprinted in *Sankhyā* **18** Parts 1 and 2, 183-194 (1957).

[21] B. Finetti, 'Initial Probabilities: a Prerequisite for any Valid Induction', *Synthèse* **20**, 2-16 (1969).

[22] For an excellent review of De Finetti's subjectivism, see M. C. Galavotti, *Subjectivism, Objectivism and Objectivity in Bruno de Finetti's Bayesianism*, https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.446.7182&rep=rep1&type=pdf.

[23] D. M. Appleby, 'Proabilities are single-case or nothing', *Optics and Spectroscopy* **99**, 447-456 (2005).

[24] T. H. Boyer, 'A Brief Survey of Stochastic Electrodynamics', *Foundations of Radiation Theory and Quantum Electrodynamics* ed. A. O. Barut, pp 49-63 Springer (1980).

[25] T. W. Marshall and E. Santos, 'Stochastic optics: A local realistic analysis of optical tests of Bell inequalities', *Physical Review A* **39**, 6271-6283 (1989).

[26] L. C. White, E. M. Pothos and J. R. Busemeyer, 'Insights from quantum cognitive models for organizational decision', *J. of Applied Research in Memory and Cognition* **4**, 229-23 (2015).

[27] W. B. Arthur, *Complexity and the Economy*, Oxford Univ. Press (2015).

[28] J. D. Farmer, 'Economics Needs to Treat the Economy as a Complex System', INET Conference Paper, (2012).

[29] T. Felin, S. Kauffman, A. Mastrogiorgio, M. Mastrogiorgio, 'Factor markets, actors, and affordances', *Industrial and Corporate Change* **25** (1), 133-147 (2016).

[30] C. Moreira and A. Wichert, 'Interference effects in quantum belief networks', *Appl. Soft Comput.* **25**, 64-85 (2014).